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On large time behavior of solutions to the compressible Navier-Stokes equation around a time periodic parallel flow

Yoshiyuki Kagei

Faculty of Mathematics, Kyushu University,
Fukuoka 819-0395, JAPAN

1 Introduction

In this article we give a summary of recent results on the stability of time-periodic parallel flows of the compressible Navier-Stokes equation in an infinite layer.

We consider the system of equations

$$\partial_{\tilde{t}}\tilde{\rho}+\operatorname{div}(\tilde{\rho}\tilde{v})=0, \tag{1.1}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v}+\tilde{v}\cdot\nabla\tilde{v})-\mu\Delta\tilde{v}-(\mu+\mu')\nabla\operatorname{div}\tilde{v}+\nabla\tilde{P}(\tilde{\rho})=\tilde{\rho}\tilde{\boldsymbol{g}}, \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell=\mathbb{R}^{n-1}\times(0,\ell)$:

$$\begin{aligned}\Omega_\ell &= \{\tilde{x}={}^T(\tilde{x}',\tilde{x}_n)\,;\\ &\qquad \tilde{x}'={}^T(\tilde{x}_1,\ldots,\tilde{x}_{n-1})\in\mathbb{R}^{n-1},\ 0<\tilde{x}_n<\ell\}.\end{aligned}$$

Here $n\geq 2$; $\tilde{\rho}=\tilde{\rho}(\tilde{x},\tilde{t})$ and $\tilde{v}={}^T(\tilde{v}^1(\tilde{x},\tilde{t}),\ldots,\tilde{v}^n(\tilde{x},\tilde{t}))$ denote the unknown density and velocity at time $\tilde{t}\geq 0$ and position $\tilde{x}\in\Omega_\ell$, respectively; $\tilde{P}$ is the pressure that is assumed to be a smooth function of $\tilde{\rho}$ satisfying $\tilde{P}'(\rho_*)>0$ for a given constant $\rho_*>0$; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu>0$, $\frac{2}{n}\mu+\mu'\geq 0$; $\operatorname{div}, \nabla$ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$, respectively. Here and in what follows ${}^T\cdot$ denotes the transposition.

Concerning the external force $\tilde{\boldsymbol{g}}$, we assume that $\tilde{\boldsymbol{g}}$ takes the form

$$\tilde{\boldsymbol{g}}={}^T(\tilde{g}^1(\tilde{x}_n,\tilde{t}),0,\ldots,0,\tilde{g}^n(\tilde{x}_n))$$

with $\tilde{g}^1(\tilde{x}_n,\tilde{t})$ being a $\tilde{T}$-periodic function in $\tilde{t}$, where $\tilde{T}>0$.

The system (1.1)–(1.2) is considered under the boundary condition

$$\widetilde{v}|_{\widetilde{x}_n=0} = \widetilde{V}^1(\widetilde{t})\boldsymbol{e}_1, \quad \widetilde{v}|_{\widetilde{x}_n=\ell} = 0, \tag{1.3}$$

and initial condition

$$(\widetilde{\rho}, \widetilde{v})|_{\widetilde{t}=0} = (\widetilde{\rho}_0, \widetilde{v}_0), \tag{1.4}$$

where $\widetilde{V}^1(\widetilde{t})$ is a $\widetilde{T}$-periodic function of $\widetilde{t}$ and $\boldsymbol{e}_1 = {}^T(1, 0, \dots, 0) \in \mathbb{R}^n$.

If $\widetilde{g}^n$ is suitably small, problem (1.1)–(1.3) has a smooth time-periodic solution $\overline{u}_p = {}^T(\overline{\rho}_p, \overline{v}_p)$, so called time-periodic parallel flow, satisfying

$$\overline{\rho}_p = \overline{\rho}_p(\widetilde{x}_n) \geq \widetilde{\underline{\rho}}, \quad \frac{1}{\ell}\int_0^\ell \overline{\rho}_p(\widetilde{x}_n)\, d\widetilde{x}_n = \rho_*,$$

$$\overline{v}_p = {}^T(\overline{v}_p^1(\widetilde{x}_n, \widetilde{t}), 0, \dots, 0), \quad \overline{v}_p^1(\widetilde{x}_n, \widetilde{t} + \widetilde{T}) = \overline{v}_p^1(\widetilde{x}_n, \widetilde{t})$$

for a positive constant $\widetilde{\underline{\rho}}$.

Our aim is to study the stability of the time-periodic parallel flow $\overline{u}_p$. We will give a summary of the results on the large time behavior of perturbations to $\overline{u}_p$ when Reynolds and Mach numbers are sufficiently small, which were recently obtained in [1, 2, 3].

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\widetilde{x} = \ell x, \ \widetilde{t} = \frac{\ell}{V} t, \ \widetilde{v} = Vv, \quad \widetilde{\rho} = \rho_*\rho, \ \widetilde{P} = \rho_* V^2 P, \ \widetilde{V}^1 = VV^1, \ \widetilde{\boldsymbol{g}} = \frac{\mu V}{\rho_* \ell^2}\boldsymbol{g}$$

with $\boldsymbol{g} = {}^T(g^1(x_n, t), \cdots, g^n(x_n))$. Here

$$\gamma = \frac{\sqrt{\widetilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu}\left\{|\partial_t \widetilde{V}^1|_{C(\mathbb{R})} + |\widetilde{g}^1|_{C(\mathbb{R}\times[0,\ell])}\right\} + |\widetilde{V}^1|_{C(\mathbb{R})} > 0.$$

Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0, 1)$; and $g^1(x_n, t)$ and $V^1(t)$ are periodic in t with period $T > 0$, where T is defined by

$$T = \frac{V}{\ell}\widetilde{T}.$$

The time-periodic parallel flow $\overline{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) \geq \underline{\rho}, \quad \int_0^1 \rho_p(x_n)\, dx_n = 1,$$

for a positive constant $\underline{\rho}$, and

$$v_p = {}^T(v_p^1(x_n, t), 0, \dots, 0), \quad v_p^1(x_n, t + T) = v_p^1(x_n, t)$$

It then follows that the perturbation $u(t) = {}^T(\phi(t), w(t)) := {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = f^0, \tag{1.5}$$

$$\begin{aligned} &\partial_t w - \tfrac{\nu}{\rho_p}\Delta w - \tfrac{\widetilde{\nu}}{\rho_p}\nabla \mathrm{div}\, w + \nabla\left(\tfrac{P'(\rho_p)}{\gamma^2 \rho_p}\phi\right) \\ &+\tfrac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)\phi \boldsymbol{e}_1 + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n \boldsymbol{e}_1 = \boldsymbol{f}, \end{aligned} \tag{1.6}$$

$$w|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0). \tag{1.8}$$

Here div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x, respectively; ν and $\widetilde{\nu}$ are the non-dimensional parameters

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \widetilde{\nu} = \nu + \nu', \quad \nu' = \frac{\mu'}{\rho_* \ell V};$$

and f^0 and $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ denote the nonlinearities:

$$f^0 = -\mathrm{div}\,(\phi w),$$

$$\begin{aligned} \boldsymbol{f} \;=\; & -w \cdot \nabla w + \tfrac{\nu\phi}{\gamma^2\rho_p^2}\left(-\Delta w + \tfrac{\partial_{x_n}^2 v_p^1}{\gamma^2\rho_p}\phi \boldsymbol{e}_1\right) - \tfrac{\nu\phi^2}{(\phi+\gamma^2\rho_p)\gamma^2\rho_p^2}\left(-\Delta w + \tfrac{\partial_{x_n}^2 v_p^1}{\gamma^2\rho_p}\phi \boldsymbol{e}_1\right) \\ & - \tfrac{\widetilde{\nu}\phi}{(\phi+\gamma^2\rho_p)\rho_p}\nabla \mathrm{div}\, w + \tfrac{\phi}{\gamma^2\rho_p}\nabla\left(\tfrac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right) - \tfrac{1}{2\gamma^4\rho_p}\nabla\left(P''(\overline{\rho}_p)\phi^2\right) \\ & + P_2(\rho_p, \phi, \partial_x \phi), \end{aligned}$$

where

$$\begin{aligned} P_2 \;=\; & \tfrac{\phi^3}{(\phi+\gamma^2\rho_p)\gamma^4\rho_p^3}\nabla P(\rho_p) - \tfrac{1}{2\gamma^4(\phi+\gamma^2\rho_p)}\nabla\left(\phi^3 P_3(\rho_p, \phi)\right) \\ & + \tfrac{\phi\nabla(P''(\rho_p)\phi^2)}{2\gamma^4\rho_p(\phi+\gamma^2\rho_p)} - \tfrac{\phi^2\nabla(P'(\rho_p)\phi)}{(\phi+\gamma^2\rho_p)\gamma^4\rho_p^2} \end{aligned}$$

with

$$P_3(\overline{\rho}_p, \phi) = \int_0^1 (1-\theta)^2 P'''(\theta\gamma^{-2}\phi + \rho_p)\, d\theta.$$

We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

As for the stability of parallel flows of the compressible Navier-Stokes equations, Iooss and Padula ([4]) studied the linearized stability of a stationary parallel flow in a cylindrical domain under the perturbations periodic in the unbounded direction of the domain. It was shown that the linearized operator generates a C_0-semigroup in L^2-space on the basic period cell with zero mean value condition for the density-component. Using the Fourier series expansion, the authors of [4] showed that the linearized semigroup is written as a direct sum of an analytic semigroup and an exponentially decaying C_0-semigroup, which correspond to low and high frequency parts of the semigroup, respectively. It was also proved that the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\{\mathrm{Re}\,\lambda = -c\}$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially.

On the other hand, the stability of a stationary parallel flow in the infinite layer Ω were considered in [5, 6, 7, 8] under the perturbations in some L^2-Sobolev space on Ω. It was shown in [5, 8] that the asymptotic leading part of the low frequency part of the linearized semigroup is given by an $n-1$ dimensional heat kernel and the high frequency part decays exponentially as $t \to \infty$, if the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to a positive constant. As for the nonlinear problem, it was proved in [5, 6, 7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$. Furthermore, the asymptotic leading part of the perturbation is given by the same $n-1$ dimensional heat kernel as in the case of the linearized problem when $n \geq 3$. In the case of $n = 2$, the asymptotic leading part is no longer described by linear heat equations but by a one-dimensional viscous Burgers equation ([7]).

These results on stationary parallel flows were extended to the time-periodic case in [1, 2, 3]. In section 2 we will give assumptions on the given data $\tilde{\boldsymbol{g}}$ and $\tilde{V}^1$ and state some properties of time-periodic parallel flow. In section 3 we will consider the linearized problem and give a summary of the results obtained in [2, 3]. We will give a Floquet representation for a part of low frequency part of the linearized evolution operator, which plays an important role in the analysis of the nonlinear problem. In section 4 we will consider the nonlinear problem and state the results on the global existence and asymptotic behavior obtained by J. Brezina ([1]).

2 Time-periodic parallel flow

We assume the following regularity for $\tilde{\boldsymbol{g}}$, $\tilde{V}^1$ and $\tilde{P}$.

Assumption 2.1 *Let m be an integer satisfying $m \geq 2$. We assume that*

$\widetilde{\boldsymbol{g}} = {}^T(\widetilde{g}^1(\widetilde{x}_n,\widetilde{t}), 0, \ldots, 0, \widetilde{g}^n(\widetilde{x}_n))$, $\widetilde{V}^1(\widetilde{t})$ *and* $\widetilde{P}$ *belong to the spaces*

$$\widetilde{g}^1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j_{per}([0,\tau]; H^{m-2j}(0,\ell)), \quad \widetilde{g}^n \in C^m[0,\ell],$$

$$\widetilde{V}^1 \in C_{per}^{[\frac{m+1}{2}]}([0,\widetilde{T}]),$$

and

$$\widetilde{P} \in C^{m+1}(\mathbb{R}).$$

It is easily verified that $\boldsymbol{g}$, V^1 and P belong to similar spaces as $\widetilde{\boldsymbol{g}}$, $\widetilde{V}^1$ and $\widetilde{P}$.

Let us consider the time-periodic parallel flow. The dimensionless form of problem (1.1)–(1.3) is written as

$$\partial_t\rho + \mathrm{div}\,(\rho v) = 0, \tag{2.1}$$

$$\rho(\partial_t v + v\cdot\nabla v) - \nu\Delta v - \widetilde{\nu}\nabla \mathrm{div}\, v + \nabla P(\rho) = \nu\rho\boldsymbol{g}, \tag{2.2}$$

$$v|_{x_n=0} = V^1(t)\boldsymbol{e}_1, \quad v|_{x_n=1} = 0. \tag{2.3}$$

The following result was shown in [2].

Proposition 2.2 ([2]) *There exists* $\delta_0 > 0$ *such that if*

$$\nu|g^n|_{C^m([0,1])} \le \delta_0,$$

then the following assertions hold true.

There exists a time-periodic solution $u_p = {}^T(\rho_p(x_n), v_p(x_n,t))$ *of* (2.1)–(2.3) *that satisfies*

$$v_p \in \bigcap_{j=0}^{[\frac{m+2}{2}]} C^j_{per}(J_T; H^{m+2-2j}(0,1)), \quad \rho_p \in C^{m+1}[0,1],$$

and

$$0 < \underline{\rho} \le \rho_p(x_n) \le \overline{\rho}, \ \int_0^1 \rho_p(x_n)dx_n = 1, \ v_p(x_n,t) = v_p^1(x_n,t)\boldsymbol{e}_1$$

with

$$P'(\rho) > 0 \ \textit{for}\ \underline{\rho} \le \rho \le \overline{\rho},$$

$$|\rho_p - 1|_{C^{m+1}([0,1])} \le \frac{C}{\gamma^2}\nu(|P''|_{C^{m-1}(\underline{\rho},\bar{\rho})} + |g^n|_{C^m([0,1])}),$$

$$|P'(\rho_p) - \gamma^2|_{C([0,1])} \le \frac{C}{\gamma^2}\nu|g^n|_{C([0,1])},$$

and

$$\frac{\rho_p P'(\rho_p)}{\gamma^2} \ge a_0 \tag{2.4}$$

for some constants $0 < \underline{\rho} < 1 < \bar{\rho}$ *and* $a_0 > 0$.

3 The linearized problem

In this section we consider the linearized problem

$$\partial_t u + L(t)u = 0, \ t > s, \ w|_{\partial\Omega} = 0, \ u|_{t=s} = u_0. \tag{3.1}$$

Here $L(t)$ is the operator given by

$$L(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_p\,\cdot) \\ \nabla\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\,\cdot\right) & -\frac{\nu}{\rho_p}\Delta I_n - \frac{\tilde{\nu}}{\rho_p}\nabla\mathrm{div} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}\partial_{x_n}^2 v_p^1(t)\boldsymbol{e}_1 & v_p^1(t)\partial_{x_1}I_n + (\partial_{x_n}v_p^1(t))\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix}.$$

Note that $L(t)$ satisfies $L(t+T) = L(t)$.

We introduce the space Z_s defined by

$$Z_s = \big\{u = {}^T(\phi, w); \phi \in C_{loc}([s,\infty); H^1(\Omega)),$$

$$\partial_{x'}^{\alpha'} w \in C_{loc}([s,\infty); L^2(\Omega)) \cap L^2_{loc}([s,\infty); H_0^1(\Omega)) \ (|\alpha'| \le 1),$$

$$w \in C_{loc}((s,\infty); H_0^1(\Omega))\big\}.$$

It was shown in [2] that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$ there exists a unique solution $u(t)$ of linear problem (3.1) in Z_s. We denote $U(t,s)$ the solution operator for (3.1) given by

$$u(t) = U(t,s)u_0.$$

To investigate problem (3.1) we consider the Fourier transform of (3.1) with respect to $x' \in \mathbb{R}^{n-1}$

$$\frac{d}{dt}\hat{u} + \hat{L}_{\xi'}(t)\hat{u} = 0, \ t > s, \ \ \hat{u}|_{t=s} = \hat{u}_0. \tag{3.2}$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is the operator on $(H^1 \times L^2)(0,1)$ defined as

$$D(\widehat{L}_{\xi'}(t)) = (H^1 \times [H^2 \cap H_0^1])(0,1),$$

$$\widehat{L}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 {\rho_p}^T \xi' & \gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\widetilde{\nu}}{\rho_p} \xi'^T \xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi' \partial_{x_n} \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2 \rho_p}\,\cdot\,\right) & -i\frac{\widetilde{\nu}}{\rho_p}{}^T \xi' \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1(t)) e_1' & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t)) e_1' \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

For each $t \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$, $\widehat{L}_{\xi'}(t)$ is sectorial on $(H^1 \times L^2)(0,1)$. We denote the solution operator for (3.2) by $\widehat{U}_{\xi'}(t,s)$. We note that it holds that

$$U(t,s)u_0 = \mathscr{F}^{-1}\left[\widehat{U}_{\xi'}(t,s)\widehat{u}_0\right]$$

for $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$.

We also need to investigate the *adjoint problem*

$$-\partial_s u + \widehat{L}_{\xi'}^*(s) u = 0, \ \ s < t, \ \ u|_{s=t} = u_0.$$

Here $\widehat{L}_{\xi'}^*(s)$ is a formal adjoint operator defined by

$$D(\widehat{L}_{\xi'}^*(s)) = (H^1 \times [H^2 \cap H_0^1])(0,1),$$

$$\widehat{L}_{\xi'}^*(s) = \begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2 {\rho_p}^T \xi' & -\gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ -i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\widetilde{\nu}}{\rho_p} \xi'^T \xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi' \partial_{x_n} \\ -\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2 \rho_p}\,\cdot\,\right) & -i\frac{\widetilde{\nu}}{\rho_p}{}^T \xi' \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial_{x_n}^2 v_p^1(s))^T e_1' & 0 \\ 0 & -i\xi_1 v_p^1(s) I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T e_1' & -i\xi_1 v_p^1(s) \end{pmatrix}.$$

We denote the solution operator for the adjoint problem by $\widehat{U}^*_{\xi'}(s,t)$.

It holds that $\widehat{U}_{\xi'}(t,s)$ and $\widehat{U}^*_{\xi'}(s,t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t,s), \ \widehat{U}^*_{\xi'}(s+T, t+T) = \widehat{U}^*_{\xi'}(s,t).$$

Since $\widehat{L}_{\xi'}(t)$ is T-periodic in t, the spectrum of $\widehat{U}_{\xi'}(T,0)$ plays an important role in the study of the large time behavior. The following results were established in [2].

We set

$$X_0 = (H^1 \times L^2)(0,1).$$

Theorem 3.1 ([2]) *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ there hold the following assertions.*

(i) The spectrum of operator $\widehat{U}_{\xi'}(T,0)$ on $(H^1 \times H^1_0)(0,1)$ satisfies

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\} \tag{3.3}$$

for a constant $q_0 > 0$ with $\frac{3}{2}q_0 < \operatorname{Re}\mu_{\xi'} < 1$. Here $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ is a simple eigenvalue of $\widehat{U}_{\xi'}(T,0)$ and $\lambda_{\xi'}$ has an expansion

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \tag{3.4}$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

Let $\widehat{\Pi}_{\xi'}$ be the eigenprojection for the eigenvalue $\mu_{\xi'}$. Then there holds

$$|\widehat{U}_{\xi'}(t,s)(I - \widehat{\Pi}_{\xi'})u|_{H^1} \leq Ce^{-d(t-s)}|(I - \widehat{\Pi}_{\xi'})u|_{X_0}$$

for $u \in X_0$ and $t - s \geq T$. Here d is a positive constant depending on r_0.

(ii) The spectrum of operator $\widehat{U}^*_{\xi'}(0,T)$ on $H^1 \times H^1_0$ satisfies

$$\sigma(\widehat{U}^*_{\xi'}(0,T)) \subset \{\overline{\mu}_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}.$$

Here $\overline{\mu}_{\xi'}$ is a simple eigenvalue of $\widehat{U}^*_{\xi'}(0,T)$.

Let $\widehat{\Pi}^*_{\xi'}$ be the eigenprojection for the eigenvalue $\overline{\mu}_{\xi'}$. Then there holds

$$\langle \widehat{\Pi}_{\xi'}u, v\rangle = \langle u, \widehat{\Pi}^*_{\xi'}v\rangle$$

for $u, v \in X_0$.

Theorem 3.1 can be proved by a perturbation argument from the case $\xi' = 0$. See [2] for details.

Based on Theorem 3.1 we can obtain a Floquet representation of a part of $U(t,s)$.

Let ν_0, γ_0 and r_0 are the numbers given by Theorem 3.1. In the rest of this section we assume that $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$.

We set

$$u^{(0)}(t) = \widehat{U}_0(t,0)u_0^{(0)}. \tag{3.5}$$

Here $u_0^{(0)}$ is an eigenfunction of the operator $\widehat{U}_0(T,0)$ for the eigenvalue $e^{\lambda_0 T} = 1$. Observe that

$$u^{(0)}(t+T) = u^{(0)}(t).$$

We also define the multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathscr{F}^{-1}\left[\widehat{\chi}_1 \lambda_{\xi'} \widehat{\sigma}\right].$$

Here $\widehat{\chi}_1$ is defined by

$$\widehat{\chi}_1(\xi') = \begin{cases} 1, & |\xi'| < r_0, \\ 0, & |\xi'| \geq r_0 \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Clearly, Λ is a bounded linear operator on $L^2(\mathbb{R}^{n-1})$. It then follows that Λ generates a uniformly continuous group $\{e^{t\Lambda}\}_{t\in\mathbb{R}}$. Furthermore, it holds that

$$\|\partial_{x'}^k e^{t\Lambda}\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},\ k = 0,1,\ldots,\ 1 \leq p \leq 2.$$

We have the following Floquet representation for $U(t,s)$.

Theorem 3.2 ([3])

(i) There exist time periodic operators

$$\mathscr{Q}(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\Omega), \quad \mathscr{Q}(t+T) = \mathscr{Q}(t),$$

$$\mathscr{P}(t) : L^2(\Omega) \to L^2(\mathbb{R}^{n-1}), \quad \mathscr{P}(t+T) = \mathscr{P}(t)$$

such that the operator $\mathbb{P}(t) := \mathscr{Q}(t)\mathscr{P}(t) : L^2(\Omega) \to L^2(\Omega)$ *satisfies*

$$\mathbb{P}(t)^2 = \mathbb{P}(t), \quad \mathbb{P}(t+t) = \mathbb{P}(t),$$

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))(\mathbb{P}(t)u(t)) = \mathscr{Q}(t)[(\partial_t - \Lambda)(\mathscr{P}(t)u(t))]$$

for $u \in L^2(0,T;(H^1 \times [H^2 \cap H_0^1])(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

(ii) It holds that

$$\mathbb{P}(t)U(t,s) = U(t,s)\mathbb{P}(s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s).$$

Furthermore,

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t,s)u\|_{L^2(\Omega)} \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_{L^1(\Omega)}$$

for $0 \leq 2j + l \leq m$, $k = 0, 1, \ldots$.

(iii) Let $\mathscr{H}(t)$ *be a heat semigroup defined by*

$$\mathscr{H}(t) = \mathscr{F}^{-1}e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)t}\mathscr{F}.$$

Suppose that $1 \leq p \leq 2$. *Then it holds that*

$$\begin{aligned}&\|\partial_{x'}^k \partial_{x_n}^l(\mathbb{P}(t)U(t,s)u - [\mathscr{H}(t-s)\sigma]u^{(0)}(t))\|_{L^2(\Omega)} \\ &\leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)}\end{aligned}$$

for $u = {}^T(\phi, w)$, $k = 0, 1, \ldots$, *and* $0 \leq l \leq m$. *Here* $u^{(0)}(t)$ *is the function given in* (3.5) *and* $\sigma = \int_0^1 \phi(x', x_n)\, dx_n$.

(iv) $(I - \mathbb{P}(t))U(t,s) = U(t,s)(I - \mathbb{P}(s))$ *satisfies*

$$\|(I - \mathbb{P}(t))U(t,s)u\|_{H^1(\Omega)} \leq Ce^{-d(t-s)}(\|u\|_{(H^1\times L^2)(\Omega)} + \|\partial_{x'}w\|_{L^2(\Omega)})$$

for $t - s \geq T$. *Here* d *is a positive constant.*

4 The nonlinear problem

In this section we consider the nonlinear problem (1.5)–(1.8).

Brezina ([1]) recently proved the global existence and the asymptotic behavior for (1.5)–(1.8) when the Reynolds and Mach numbers are sufficiently small.

Theorem 4.1 ([1]) *Let* $n \geq 2$ *and let* m *be an integer satisfying* $m \geq [n/2]+1$. *Suppose that* $\tilde{\boldsymbol{g}}$, $\tilde{V}^1$ *and* $\tilde{P}$ *satisfy Assumption 2.1 for* m *replaced by* $m+1$. *Then there are positive numbers* ν_1 *and* γ_1 *such that the following assertions hold true, provided that* $\nu \geq \nu_1$ *and* $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_1^2$.

There is a positive number ε_0 such that if $u_0 \in {}^T(\phi_0, w_0) \in H^m \cap L^1(\Omega)$ satisfies a suitable compatibility condition and $\|u_0\|_{H^m \cap L^1(\Omega)} \le \varepsilon_0$, then there exists a global solution $u(t)$ of (1.5)–(1.8) in $C([0,\infty); H^m(\Omega))$ and $u(t)$ satisfies

$$\|\partial_{x'}^k u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \quad k = 0, 1,$$

as $t \to \infty$.

Furthermore, there holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t))$$

as $t \to \infty$. Here $\eta_n(t) = 1$ for $n \ge 4$, $\eta_n(t) = \log t$ for $n = 3$ and $\eta_n(t) = t^\delta$ for $n = 2$, where δ is an arbitrarily positive number; $u^{(0)} = u^{(0)}(x_n, t)$ is the function given in (3.5); and $\sigma = \sigma(x', t)$ satisfies

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n$$

if $n \ge 3$, and

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + a_0 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n$$

if $n = 2$, where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{n-1}^2$ for $n \ge 3$, and a_0 is a constant.

Remark 4.2 A result similar to Theorem 4.1 also holds for the case of stationary parallel flows ([7]).

Theorem 4.1 is proved by the decomposition method based on the spectral analysis in section 3. We write problem (1.5)–(1.8) as

$$\partial_t u + L(t)u = \boldsymbol{F}(u), \quad u(0) = u_0.$$

We decompose the solution $u(t)$ of (1.5)–(1.8) into

$$u(t) = u_1(t) + u_\infty(t),$$

where

$$u_1(t) = \mathbb{P}(t)u(t), \quad u_\infty(t) = (I - \mathbb{P}(t))u(t).$$

It then follows from Theorem 3.2 that

$$u_1(t) = \mathscr{Q}(t)\left[e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-s)\Lambda}\mathscr{P}(s)\boldsymbol{F}(u(s))\, ds\right],$$

$$\partial_t u_\infty + L(t)u_\infty = (I - \mathbb{P}(t))\boldsymbol{F}(u), \quad u_\infty(0) = (I - \mathbb{P}(t))u_0.$$

To estimate u_1, we use the estimates obtained in Theorem 3.2, while u_∞ is estimated by a variant of the Matsumura-Nishida energy method ([9, 6, 7]). See [1] for details.

References

[1] Brezina, J. Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow. *MI Preprint Series, Kyushu University* 2012-10.

[2] Brezina, J., Kagei, Y. (2012). Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow. *Mathematical Models and Methods in Applied Sciences* **22**, pp. 1250007-1-1250007-53.

[3] Brezina, J., Kagei, Y. (2013). Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow. *J. Differential Equations* **255**, pp. 1132-1195.

[4] Iooss, G., Padula, M. (1998). Structure of the linearized problem for compressible parallel fluid flows. *Ann. Univ. Ferrara*, Sez. VII **43**, pp. 157-171.

[5] Kagei, Y., (2011). Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow. *J. Math. Fluid Mech.* **13**, pp. 1–31.

[6] Kagei, Y. (2011). Global existence of solutions to the compressible Navier-Stokes equation around parallel flows. *J. Differential Equations* **251**, pp. 3248-3295.

[7] Kagei, Y. (2012). Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow. *Arch. Rational Mech. Anal.* **205**, pp. 585–650.

[8] Kagei, Y., Nagafuchi, Y., Sudou, T. (2010). Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow. *J. Math-for-Ind.* **2A**, pp. 39-56. Correction to "Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow" in *J. Math-for-Ind.* **2A** (2010), pp. 39-56 *J. Math-for-Ind.* **2B** (2010), pp. 235.

[9] Matsumura, A., Nishida, T. (1983). Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.* **89**, pp. 445–464.